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An Extended Trapezoidal Formula for the Diffusion Equation in Two Space Dimensions

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Abstract—We describe a *locally one-dimensional* (LOD) time integration scheme for the diffusion equation in two space dimensions: $u_t = \nu(u_{xx} + u_{yy})$, based on the *extended trapezoidal formula* (ETF). The resulting LOD-ETF scheme is *third order* in time and is *unconditionally stable*. We describe the scheme for both Dirichlet and Neumann boundary conditions. We then extend the LOD-ETF scheme for nonlinear reaction-diffusion equations and for the convection-diffusion equation in two space dimensions. Numerical experiments are given to illustrate the obtained scheme and to compare its performance with the better-known LOD Crank-Nicolson scheme. While the LOD Crank-Nicolson scheme can give unwanted oscillations in the computed solution, our present LOD-ETF scheme provides both stable and accurate approximations for the true solution.

Keywords—Diffusion equation, Two space dimensions, Extended trapezoidal formula, Unconditional stability, Crank-Nicolson scheme, Nonlinear reaction-diffusion equations.

1. INTRODUCTION

We consider the diffusion equation in two space dimensions

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad 0 < x < \ell_x, \quad 0 < x < \ell_y, \quad t > 0, \tag{1.1}$$

subject to the initial condition

$$u(x,y,0) = f(x,y), \qquad 0 \le x \le \ell_x, \quad 0 \le y \le \ell_y, \tag{1.1a}$$

and the Dirichlet boundary conditions:

$$\left.\begin{aligned} u(0,y,t) &= g_0(y,t), & u(\ell_x,y,t) &= g_1(y,t),\\ u(x,0,t) &= h_0(x,t), & u(x,\ell_y,t) &= h_1(x,t), \end{aligned}\right\} \qquad t \ge 0. \tag{1.1b}$$

For a positive integer N, we consider the rectangular spatial grid (x_i, y_j), $x_i = i\Delta x$, $y_j = j\Delta y$, $i,j = 0(1)N+1$, with spatial increments $\Delta x = \ell_x/(N+1)$, $\Delta y = \ell_y/(N+1)$, and let $t_n = n\Delta t$,

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$n = 0, 1, \ldots$, for temporal increment $\Delta t > 0$. In the following, we set $r_x = \nu\Delta t/\Delta x^2$, $r_y = \nu\Delta t/\Delta y^2$, and $u^n_{i,j} = u(x_i, y_j, t_n)$, etc.

The better-known LOD finite difference schemes for the diffusion equation in two space dimensions are those which employ, for integration in time, the backward Euler formula and the classical trapezoidal formula; see, for example, [1–3]. The latter scheme, often called (after Crank and Nicolson [4]) the LOD Crank-Nicolson scheme, is described by the pair of equations

$$\left(1 - \frac{r_x}{2}\,\delta_x^2\right) u^{n+1/2}_{i,j} = \left(1 + \frac{r_x}{2}\,\delta_x^2\right) u^n_{i,j}, \tag{1.2a}$$

$$\left(1 - \frac{r_y}{2}\,\delta_y^2\right) u^{n+1}_{i,j} = \left(1 + \frac{r_y}{2}\,\delta_y^2\right) u^{n+1/2}_{i,j}. \tag{1.2b}$$

This scheme is second order in time and is unconditionally stable. A modification of scheme (1.2) which is fourth order in space is described by Mitchell and Griffiths [4]; their scheme employs Numerov discretization for the spatial derivatives.

Recently, Chawla and Al-Zanaidi [5] have described a third-order unconditionally stable time integration scheme for the diffusion equation in one space dimension based on the *extended trapezoidal formula* of Usmani and Agarwal [6]; see the Appendix in [5]. It may be noted here that an *alternating direction implicit* (ADI) implementation of the extended trapezoidal formula does not produce an unconditionally stable scheme for the diffusion equation in two space dimensions.

In the present paper, we describe a *locally one-dimensional* (LOD) time integration scheme for the diffusion equation in two space dimensions (1.1) based on the extended trapezoidal formula (ETF). The resulting LOD-ETF scheme is *third order* in time and is *unconditionally stable*. We describe the scheme for both Dirichlet and Neumann boundary conditions. We then extend the LOD-ETF scheme for nonlinear reaction-diffusion equations and for the convection-diffusion equation in two space dimensions. Numerical experiments are given to illustrate the obtained scheme and to compare its performance with the better-known LOD Crank-Nicolson scheme. While the LOD Crank-Nicolson scheme can give unwanted oscillations in the computed solution, our present LOD-ETF scheme provides both stable and accurate approximations for the true solution.

2. LOD EXTENDED TRAPEZOIDAL FORMULA SCHEME

Discretizing the spatial derivative in $u_t = 2\nu u_{xx}$ by the central difference formula, we obtain for $j = 1, \ldots, N$,

$$\frac{\partial}{\partial t} u_{i,j}(t) = \frac{2\nu}{\Delta x^2}\left[u_{i+1,j}(t) - 2u_{i,j}(t) + u_{i-1,j}(t)\right], \qquad i = 1, \ldots, N. \tag{2.1}$$

Let

$$\mathbf{u}_j(t) = \begin{bmatrix} u_{1,j}(t) \\ \cdot \\ \cdot \\ \cdot \\ u_{N,j}(t) \end{bmatrix}, \quad J = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \mathbf{a}_j(t) = \begin{bmatrix} u_{0,j}(t) \\ 0 \\ \cdot \\ 0 \\ u_{N+1,j}(t) \end{bmatrix}.$$

Then using the boundary conditions, we can write system (2.1) as

$$\frac{\partial}{\partial t} \mathbf{u}_j(t) = \frac{2\nu}{\Delta x^2}\left\{\mathbf{a}_j(t) - J\mathbf{u}_j(t)\right\}, \qquad j = 1, \ldots, N, \tag{2.2}$$

with the initial condition: $\mathbf{u}_j(0) = [f(x_1, y_j), \ldots, f(x_N, y_j)]^\top$.

Now, applying the *extended trapezoidal formula* (ETF) for the time integration of (2.2) from t_n to $t_{n+1/2}$, we have

$$\hat{\mathbf{u}}^{n+1}_j = \mathbf{u}^n_j - 2r_x J\mathbf{u}^{n+1/2}_j + 2r_x \mathbf{a}^{n+1/2}_j \tag{2.3}$$

and

$$\begin{aligned}\mathbf{u}_j^{n+1/2} = \mathbf{u}_j^n - \frac{r_x}{12} J \left[5\mathbf{u}_j^n + 8\mathbf{u}_j^{n+1/2} - \hat{\mathbf{u}}_j^{n+1}\right] \\ + \frac{r_x}{12}\left[5\mathbf{a}_j^n + 8\mathbf{a}_j^{n+1/2} - \mathbf{a}_j^{n+1}\right].\end{aligned} \tag{2.4}$$

Substituting for $\hat{\mathbf{u}}_j^{n+1}$ from (2.3) in (2.4), we obtain for $j = 1, \dots, N$,

$$\begin{aligned}\left[I + \frac{2}{3} r_x J + \frac{1}{6}(r_x J)^2\right] \mathbf{u}_j^{n+1/2} = \left[I - \frac{1}{3} r_x J\right] \mathbf{u}_j^n \\ + \frac{5}{12} r_x \mathbf{a}_j^n + \frac{r_x}{6}\left[4I + r_x J\right] \mathbf{a}_j^{n+1/2} - \frac{r_x}{12} \mathbf{a}_j^{n+1}\end{aligned} \tag{2.5}$$

(I denotes identity matrix).

Again, discretizing the spatial derivative in $u_t = 2\nu u_{yy}$ by the central difference formula, we obtain for $i = 1, \dots, N$,

$$\frac{\partial}{\partial t} u_{i,j}(t) = \frac{2\nu}{\Delta y^2}\left[u_{i,j+1}(t) - 2u_{i,j}(t) + u_{i,j-1}(t)\right], \qquad j = 1, \dots, N. \tag{2.6}$$

Now, let

$$\mathbf{v}_i(t) = \begin{bmatrix} u_{i,1}(t) \\ \cdot \\ \cdot \\ \cdot \\ u_{i,N}(t) \end{bmatrix}, \qquad \mathbf{b}_i(t) = \begin{bmatrix} u_{i,0}(t) \\ 0 \\ \cdot \\ 0 \\ u_{i,N+1}(t) \end{bmatrix}.$$

Then using the boundary conditions, we can write system (2.6) as

$$\frac{\partial}{\partial t} \mathbf{v}_i(t) = \frac{2\nu}{\Delta y^2}\left\{\mathbf{b}_i(t) - J\mathbf{v}_i(t)\right\}, \qquad i = 1, \dots, N. \tag{2.7}$$

with the initial condition $\mathbf{v}_i(0) = [f(x_i, y_1), \dots, f(x_i, y_N)]^{\top}$.

Now applying the *extended trapezoidal formula* (ETF) for the time integration of (2.7) from $t_{n+1/2}$ to t_{n+1}, we have

$$\hat{\mathbf{v}}_i^{n+3/2} = \mathbf{v}_i^{n+1/2} - 2r_y J \mathbf{v}_i^{n+1} + 2r_y \mathbf{b}_i^{n+1} \tag{2.8}$$

and

$$\begin{aligned}\mathbf{v}_i^{n+1} = \mathbf{v}_i^{n+1/2} - \frac{r_y}{12} J \left[5\mathbf{v}_i^{n+1/2} + 8\mathbf{v}_i^{n+1} - \hat{\mathbf{v}}_i^{n+3/2}\right] \\ + \frac{r_y}{12}\left[5\mathbf{b}_i^{n+1/2} + 8\mathbf{b}_i^{n+1} - \mathbf{b}_i^{n+3/2}\right].\end{aligned} \tag{2.9}$$

Substituting for $\hat{\mathbf{v}}_i^{n+3/2}$ from (2.8) in (2.9), we obtain for $i = 1, \dots, N$,

$$\begin{aligned}\left[I + \frac{2}{3} r_y J + \frac{1}{6}(r_y J)^2\right] \mathbf{v}_i^{n+1} = \left[I - \frac{1}{3} r_y J\right] \mathbf{v}_i^{n+1/2} \\ + \frac{5}{12} r_y \mathbf{b}_i^{n+1/2} + \frac{r_y}{6}\left[4I + r_y J\right] \mathbf{b}_i^{n+1} - \frac{r_y}{12} \mathbf{b}_i^{n+3/2}.\end{aligned} \tag{2.10}$$

We call the scheme described by the pair of equations (2.5) and (2.10) a *locally one-dimensional extended trapezoidal formula* (LOD-ETF) scheme for the diffusion equation (1.1).

Note that for the matrix of unknowns

$$U = [u_{i,j}]_{i,j=1}^N = [\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N] = \left[\mathbf{v}_1^{\top}, \mathbf{v}_2^{\top}, \dots \mathbf{v}_N^{\top}\right]^{\top},$$

the scheme computes the column vectors $\mathbf{u}_j$ for time level $t_{n+1/2}$ by (2.5) followed by the computation of the row vectors $\mathbf{v}_i^{\top}$ for time level t_{n+1} by (2.10). In each case, it involves the solution of a pentadiagonal linear system.

Since the extended trapezoidal formula is third order in time, it is clear that each time-stage of the scheme, (2.5) followed by (2.10), and hence, the overall scheme from time level t_n to t_{n+1} is *third order* in time.

2.1. Stability of the LOD-ETF Scheme

We introduce

$$A_x = I + \frac{2}{3} r_x J + \frac{1}{6} (r_x J)^2, \qquad B_x = I - \frac{1}{3} r_x J,$$

with A_y, B_y defined similarly. Then, for homogeneous boundary conditions, the LOD-ETF scheme described by (2.5) and (2.10) can be written as

$$A_x U^{n+1/2} = B_x U^n, \tag{2.11a}$$

$$A_y U^{n+1} = B_y U^{n+1/2}, \tag{2.11b}$$

and hence,

$$U^{n+1} = QU^n, \tag{2.12}$$

where the *amplification matrix* Q is given by

$$Q = \left(A_y^{-1} B_y\right) \left(A_x^{-1} B_x\right). \tag{2.13}$$

The matrix J has N distinct and positive eigenvalues given by (see, for example, [7]), $\lambda_s = 4\sin^2((s\pi)/(2(N+1)))$, $s = 1, \dots, N$. For an eigenvalue λ of J, setting setting $\sigma_x = \lambda r_x$, the corresponding eigenvalue μ_x of $A_x^{-1} B_x$ is given by

$$\mu_x = \frac{1 - (1/3)\,\sigma_x}{1 + (2/3)\,\sigma_x + (1/6)\,\sigma_x^2}. \tag{2.14}$$

It is easy to see that $|\mu_x| \le 1$. Likewise for an eigenvalue μ_y of $A_y^{-1} B_y$, $|\mu_y| \le 1$. If now μ denotes an eigenvalue of Q, since all the matrices involved here commute and have the same set of eigenvectors, therefore, $\mu = \mu_x \mu_y$, and it follows that $|\mu| \le 1$. Hence, the LOD-ETF scheme is unconditionally stable.

3. LOD-ETF SCHEME FOR NEUMANN BOUNDARY CONDITIONS

There are several possibilities in which Neumann boundary conditions can be specified. To illustrate the extension of the above LOD-ETF scheme, we consider Neumann boundary conditions specified as follows.

$$\begin{aligned} \frac{\partial}{\partial x} u(0, y, t) &= a_1 u(0, y, t) - b_1, \\ \frac{\partial}{\partial x} u(\ell_x, y, t) &= -a_2 u(\ell_x, y, t) + b_2, \end{aligned} \tag{3.1}$$

where a_1, b_1, a_2, b_2 are nonnegative constants. For this case, it is helpful to redefine the x-grid by $x_i = (i-1)\Delta x$, $i = 1, \dots, N$, where $\Delta x = \ell_x/(N-1)$. We first consider discretization of the boundary condition along the line $x_1 = 0$. Introducing the 'fictitious line' $x = x_0$, and replacing the derivative by the central difference approximation, we have

$$\frac{1}{2\Delta x} \left[u_{2,j}(t) - u_{0,j}(t)\right] = a_1 u_{1,j}(t) - b_1. \tag{3.2}$$

With the help of (3.2), eliminating $u_{0,j}(t)$ from the discretization equation (2.1) for $i = 0$, we obtain

$$\frac{\partial}{\partial t} u_{1,j}(t) = \frac{2v}{\Delta x^2} \left[2u_{2,j} - 2(1 + \Delta x a_1) u_{1,j} + 2\Delta x b_1\right]. \tag{3.3}$$

In an analogous manner, for the boundary condition at $x = \ell_x$, we obtain

$$\frac{\partial}{\partial t} u_{N,j}(t) = \frac{2\nu}{\Delta x^2} \left[2u_{N-1,j} - 2(1 + \Delta x a_2) u_{N,j} + 2\Delta x b_2\right]. \tag{3.4}$$

Now, let

$$M = \begin{bmatrix} 2(1+\Delta x a_1) & -2 & & & \\ -1 & 2 & -1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 2 & -1 \\ & & & -2 & 2(1+\Delta x a_2) \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 2\Delta x b_1 \\ 0 \\ \cdot \\ 0 \\ 2\Delta x b_2 \end{bmatrix},$$

Then, the spatial discretizations (3.3), (2.1) for $i = 2(1)N-1$ and (3.4) can be written as

$$\frac{\partial}{\partial t}\mathbf{u}_j(t) = \frac{2v}{\Delta x^2}\{\mathbf{c} - M\mathbf{u}_j(t)\}, \qquad j = 1,\dots,N. \tag{3.5}$$

The second equation (2.7) remains the same as before. Now, applying the extended trapezoidal formula for the time integration of (3.5) from t_n to $t_{n+1/2}$, we obtain

$$\hat{\mathbf{u}}_j^{n+1} = \mathbf{u}_j^n - 2r_x M\mathbf{u}_j^{n+1/2} + 2r_x\mathbf{c} \tag{3.6}$$

and

$$\mathbf{u}_j^{n+1/2} = \mathbf{u}_j^n - \frac{r_x}{12}M\left[5\mathbf{u}_j^n + 8\mathbf{u}_j^{n+1/2} - \hat{\mathbf{u}}_j^{n+1}\right] + r_x\mathbf{c}. \tag{3.7}$$

Substituting for $\hat{\mathbf{u}}_j^{n+1}$ from (3.6) in (3.7), we obtain for $j = 1,\dots,N$,

$$\left[I + \frac{2}{3}r_x M + \frac{1}{6}(r_x M)^2\right]\mathbf{u}_j^{n+1/2} = \left[I - \frac{1}{3}r_x M\right]\mathbf{u}_j^n + r_x\left[I + \frac{1}{6}r_x M\right]\mathbf{c}. \tag{3.8}$$

The pair of equations (3.8) and (2.10) describe the LOD-ETF scheme for the diffusion equation (1.1) for the case of Neumann boundary conditions (3.1).

4. LOD-ETF SCHEME FOR NONLINEAR REACTION-DIFFUSION EQUATIONS

We next extend the LOD-ETF scheme for nonlinear reaction-diffusion equations.

$$\frac{\partial u}{\partial t} = v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + p(u). \tag{4.1}$$

Discretizing the spatial derivative in $u_t = 2\nu u_{xx} + 2p(u)$ by the central difference formula, and setting

$$\mathbf{p}(\mathbf{u}_j(t)) = [p(u_{1,j}(t)),\dots,p(u_{N,j}(t))]^{\mathsf{T}},$$

we obtain

$$\frac{\partial}{\partial t}\mathbf{u}_j(t) = \frac{2\nu}{\Delta x^2}\{\mathbf{a}_j(t) - J\mathbf{u}_j(t)\} + 2\mathbf{p}(\mathbf{u}_j(t)), \qquad j = 1,\dots,N. \tag{4.2}$$

Now, applying the extended trapezoidal formula for the time integration of (4.2) from t_n to $t_{n+1/2}$, we have

$$\hat{\mathbf{u}}_j^{n+1} = \mathbf{u}_j^n - 2r_x J\mathbf{u}_j^{n+1/2} + 2r_x\mathbf{a}_j^{n+1/2} + 2\Delta t\,\mathbf{p}\left(\mathbf{u}_j^{n+1/2}\right) \tag{4.3}$$

and

$$\begin{aligned}\mathbf{u}_j^{n+1/2} = \mathbf{u}_j^n &- \frac{r_x}{12}J\left[5\mathbf{u}_j^n + 8\mathbf{u}_j^{n+1/2} - \hat{\mathbf{u}}_j^{n+1}\right] + \frac{r_x}{12}\left[5\mathbf{a}_j^n + 8\mathbf{a}_j^{n+1/2} - \mathbf{a}_j^{n+1}\right] \\ &+ \frac{\Delta t}{12}\left[5\mathbf{p}\left(\mathbf{u}_j^n\right) + 8\mathbf{p}\left(\mathbf{u}_j^{n+1/2}\right) - \mathbf{p}\left(\hat{\mathbf{u}}_j^{n+1}\right)\right].\end{aligned} \tag{4.4}$$

Substituting for $\hat{\mathbf{u}}_j^{n+1}$ from (4.3) in the linear part of (4.4), we obtain for $j = 1, \dots, N$,

$$\begin{aligned}\left[I + \frac{2}{3} r_x J + \frac{1}{6} (r_x J)^2\right] \mathbf{u}_j^{n+1/2} = {} & \left[I - \frac{1}{3} r_x J\right] \mathbf{u}_j^n \\ & + \frac{5}{12} r_x \mathbf{a}_j^n + \frac{r_x}{6} [4I + r_x J] \mathbf{a}_j^{n+1/2} - \frac{r_x}{12} \mathbf{a}_j^{n+1} \\ & + \frac{5\Delta t}{12} \mathbf{p}\left(\mathbf{u}_j^n\right) + \frac{\Delta t}{6} [4I + r_x J] \mathbf{p}\left(\mathbf{u}_j^{n+1/2}\right) \\ & - \frac{\Delta t}{12} \mathbf{p}\left(\hat{\mathbf{u}}_j^{n+1}\right).\end{aligned} \tag{4.5}$$

Again, discretizing the spatial derivative in $u_t = 2\nu u_{yy} + 2p(u)$ by the central difference formula, and setting

$$\mathbf{p}(\mathbf{v}_i(t)) = [p(u_{i,1}(t)), \dots, p(u_{i,N}(t))]^\top,$$

we obtain

$$\frac{\partial}{\partial t} \mathbf{v}_i(t) = \frac{2\nu}{\Delta y^2} \{\mathbf{b}_i(t) - J\mathbf{v}_i(t)\} + 2\mathbf{p}(\mathbf{v}_i(t)), \qquad i = 1, \dots, N. \tag{4.6}$$

Applying the extended trapezoidal formula for the time integration of (4.6) from $t_{n+1/2}$ to t_{n+1}, we have

$$\hat{\mathbf{v}}_i^{n+3/2} = \mathbf{v}_i^{n+1/2} - 2r_y J \mathbf{v}_i^{n+1} + 2r_y \mathbf{b}_i^{n+1} + 2\Delta t \mathbf{p}\left(\mathbf{v}_i^{n+1}\right) \tag{4.7}$$

and

$$\begin{aligned}\mathbf{v}_i^{n+1} = {} & \mathbf{v}_i^{n+1/2} - \frac{r_y}{12} J \left[5\mathbf{v}_i^{n+1/2} + 8\mathbf{v}_i^{n+1} - \hat{\mathbf{v}}_i^{n+3/2}\right] \\ & + \frac{r_y}{12} \left[5\mathbf{b}_i^{n+1/2} + 8\mathbf{b}_i^{n+1} - \mathbf{b}_i^{n+3/2}\right] \\ & + \frac{\Delta t}{12} \left[5\mathbf{p}\left(\mathbf{v}_i^{n+1/2}\right) + 8\mathbf{p}\left(\mathbf{v}_i^{n+1}\right) - \mathbf{p}\left(\hat{\mathbf{v}}_i^{n+3/2}\right)\right].\end{aligned} \tag{4.8}$$

Substituting for $\hat{\mathbf{v}}_i^{n+3/2}$ from (4.7) in the linear part of (4.8), we obtain $i = 1, \dots, N$,

$$\begin{aligned}\left[I + \frac{2}{3} r_y J + \frac{1}{6} (r_y J)^2\right] \mathbf{v}_i^{n+1} = {} & \left[I - \frac{1}{3} r_y J\right] \mathbf{v}_i^{n+1/2} \\ & + \frac{5}{12} r_y \mathbf{b}_i^{n+1/2} + \frac{r_y}{6} (4I + r_y J) \mathbf{b}_i^{n+1} - \frac{r_y}{12} b_i^{n+3/2} \\ & + \frac{5\Delta t}{12} \mathbf{p}\left(\mathbf{v}_i^{n+1/2}\right) + \frac{\Delta t}{6} (4I + r_y J) \mathbf{p}\left(\mathbf{v}_i^{n+1}\right) \\ & - \frac{\Delta t}{12} \mathbf{p}\left(\hat{\mathbf{v}}_i^{n+3/2}\right).\end{aligned} \tag{4.9}$$

The pair of equations (4.5) and (4.9) describes the LOD-ETF scheme for the nonlinear reaction-diffusion equation (4.1). The nonlinear system in each case can be solved by Newton's method by taking, for example, as initial approximation the value of the solution at the previous time level. Note that the Jacobian of the system in each case is pentadiagonal.

5. LOD-ETF SCHEME FOR THE CONVECTION-DIFFUSION EQUATION

We next consider an extension of the extended trapezoidal formula scheme for the convection-diffusion equation in two space dimensions:

$$\frac{\partial u}{\partial t} + c_1 \frac{\partial u}{\partial x} + c_2 \frac{\partial u}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{5.1}$$

with initial and boundary conditions given in (1.1a) and (1.1b). Extension of the scheme presented below for the case of Neumann boundary conditions can be done as indicated in Section 3 above.

Before we describe the finite difference scheme, it might be of interest to note here that through the transformation

$$v(x,y,t) = \exp\left(\frac{1}{2v}(c_1x + c_2y)\right)\exp\left(-\frac{1}{4v}\left(c_1^2 + c_2^2\right)t\right)u(x,y,t), \tag{5.2}$$

the convection-diffusion equation (5.1), given in v, can be transformed into the (pure) diffusion equation: $u_t = v(u_{xx}+u_{yy})$. Together with the correspondingly transformed initial condition and the boundary conditions, the equation can thus be integrated by the LOD-ETF scheme described in Section 2.

We now proceed to describe an extended trapezoidal scheme for the direction integration of the convection-diffusion equation (5.1). Discretizing the spatial derivative in $u_t + 2c_1u_x = 2\nu u_{xx}$ by the central difference formula, we obtain for $j = 1, \dots, N$:

$$\begin{gathered}\frac{\partial}{\partial t}u_{i,j}(t) + \frac{c_1}{\Delta x}\left[u_{i+1,j} - u_{i-1,j}\right] = \frac{2\nu}{\Delta x^2}\left[u_{i+1,j}(t) - 2u_{i,j}(t) + u_{i-1,j}(t)\right],\\ i = 1, \dots, N.\end{gathered} \tag{5.3}$$

Now, let

$$\mathbf{b}_j(t) = \begin{bmatrix} u_{0,j}(t) \\ 0 \\ \cdot \\ 0 \\ -u_{N+1,j}(t) \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}.$$

Then using the boundary conditions, we can write system (5.3) as

$$\frac{\partial}{\partial t}\mathbf{u}_j(t) = \frac{2\nu}{\Delta x^2}\left\{\mathbf{a}_j(t) - J\mathbf{u}_j(t)\right\} + \frac{c_1}{\Delta x}\left\{\mathbf{b}_j - B\mathbf{u}_j\right\}, \qquad j = 1, \dots, N. \tag{5.4}$$

Now, applying the extended trapezoidal formula for the time integration of (5.4) from t_n to $t_{n+1/2}$, we have

$$\hat{\mathbf{u}}_j^{n+1} = \mathbf{u}_j^n - 2C_x\mathbf{u}_j^{n+1/2} + 2\mathbf{c}_j^{n+1/2} \tag{5.5}$$

and

$$\begin{aligned}\mathbf{u}_j^{n+1/2} = \mathbf{u}_j^n &- \frac{1}{12}C_x\left[5\mathbf{u}_j^n + 8\mathbf{u}_j^{n+1/2} - \hat{\mathbf{u}}_j^{n+1}\right]\\ &+ \frac{1}{12}\left[5\mathbf{c}_j^n + 8\mathbf{c}_j^{n+1/2} - \mathbf{c}_j^{n+1}\right],\end{aligned} \tag{5.6}$$

where we have set $\rho_x = c_1\Delta t/\Delta x$, and

$$\mathbf{c}_j(t) = r_x\mathbf{a}_j(t) + \frac{\rho x}{2}\mathbf{b}_j(t), \qquad C_x = r_xJ + \frac{\rho_x}{2}B.$$

Substituting for $\hat{\mathbf{u}}_j^{n+1}$ from (5.5) in (5.6), we obtain for $j = 1, \dots, N$,

$$\begin{aligned}\left[I + \frac{2}{3}C_x + \frac{1}{6}C_x^2\right]\mathbf{u}_j^{n+1/2} &= \left[I - \frac{1}{3}C_x\right]\mathbf{u}_j^n\\ &+\frac{5}{12}\mathbf{c}_j^n + \frac{1}{6}\left[4I + C_x\right]\mathbf{c}_j^{n+1/2} - \frac{1}{12}\mathbf{c}_j^{n+1}.\end{aligned} \tag{5.7}$$

Again, discretizing the spatial derivative in $u_t+2c_2u_y = 2\nu u_{yy}$ by the central difference formula, we have $i = 1, \dots, N$,

$$\begin{gathered}\frac{\partial}{\partial t}u_{i,j}(t) + \frac{c_2}{\Delta y}\left[u_{i,j+1} - u_{i,j-1}\right] = \frac{2\nu}{\Delta y^2}\left[u_{i,j+1}(t) - 2u_{i,j}(t) + u_{i,j-1}(t)\right],\\ j = 1, \dots, N.\end{gathered} \tag{5.8}$$

Let now

$$\mathbf{d}_i(t) = \begin{bmatrix} u_{i,0}(t) \\ 0 \\ \cdot \\ 0 \\ u_{i,N+1}(t) \end{bmatrix}, \qquad \mathbf{e}_i(t) = \begin{bmatrix} u_{i,0}(t) \\ 0 \\ \cdot \\ 0 \\ -u_{i,N+1}(t) \end{bmatrix}.$$

Then using the boundary conditions, we can write system (5.8) as

$$\frac{\partial}{\partial t}\,\mathbf{v}_i(t) = \frac{2\nu}{\Delta y^2}\,\{\mathbf{d}_i(t) - J\mathbf{v}_i(t)\} + \frac{c_2}{\Delta y}\,\{\mathbf{e}_i - B\mathbf{v}_i\}, \qquad i = 1,\dots,N. \tag{5.9}$$

Applying the extended trapezoidal formula for the time integration of (5.9) from $t_{n+1/2}$ to t_{n+1}, we have

$$\hat{\mathbf{v}}_i^{n+3/2} = \mathbf{v}_i^{n+1/2} - 2C_y\mathbf{v}_i^{n+1} + 2\mathbf{f}_i^{n+1} \tag{5.10}$$

and

$$\begin{aligned} \mathbf{v}_i^{n+1} = \mathbf{v}_i^{n+1/2} &- \frac{1}{12}\,C_y\left[5\mathbf{v}_i^{n+1/2} + 8\mathbf{v}_i^{n+1} - \hat{\mathbf{v}}_i^{n+3/2}\right] \\ &+ \frac{1}{12}\left[5\mathbf{f}_i^{n+1/2} + 8\mathbf{f}_i^{n+1} - \mathbf{f}_i^{n+3/2}\right], \end{aligned} \tag{5.11}$$

where we have set $\rho_y = c_2\Delta t/\Delta y$, and

$$\mathbf{f}_i(t) = r_y\mathbf{d}_i(t) + \frac{\rho_y}{2}\,\mathbf{e}_i(t), \qquad C_y = r_yJ + \frac{\rho_y}{2}\,B.$$

Substituting for $\hat{\mathbf{v}}_i^{n+3/2}$ from (5.10) in (5.11), we obtain for $i = 1,\dots,N$,

$$\begin{aligned} \left[I + \frac{2}{3}\,C_y + \frac{1}{6}\,C_y^2\right]\mathbf{v}_i^{n+1} &= \left[I - \frac{1}{3}\,C_y\right]\mathbf{v}_i^{n+1/2} \\ &+ \frac{5}{12}\,\mathbf{f}_i^{n+1/2} + \frac{1}{6}\,[4I + C_y]\,\mathbf{f}_i^{n+1} - \frac{1}{12}\,\mathbf{f}_i^{n+3/2}. \end{aligned} \tag{5.12}$$

Equations (5.7) and (5.12) describe the LOD-ETF scheme for the convection-diffusion equation (5.1).

5.1. Stability of the LOD-ETF Scheme

Let δ_x, Δ_x, and ∇_x denote, respectively, central, forward, and backward differences. Then, it is easy to see the correspondence

$$C_x \to -r_x\delta_x^2 + \frac{\rho_x}{2}\,(\Delta_x + \nabla_x).$$

Consider the Fourier mode

$$u_{i,j}^n = \tilde{u}^n \exp\left(\sqrt{-1}\,(\beta_x x_i + \beta_y y_j)\right), \qquad \beta_x,\ \beta_y > 0. \tag{5.13}$$

It then follows that

$$C_x u_{i,j}^n \to -z_x u_{i,j}^n, \tag{5.14}$$

where we have set

$$\xi_x = \beta_x\Delta x, \qquad a_x = 4r_x\sin^2\left(\frac{\xi_x}{2}\right), \qquad b_x = \rho_x\sin(\xi_x), \qquad z_x = -a_x + \sqrt{-1}\,b_x.$$

Now, writing the scheme (5.7) in scalar form, and applying it to the Fourier mode (5.13), in view of (5.14), we obtain

$$\tilde{u}^{n+1/2} = s_x\tilde{u}^n, \tag{5.15}$$

where the *symbol* of the scheme s_x is given by

$$s_x = \frac{1 + (1/3)\,z_x}{1 - (2/3)\,z_x + (1/6)\,z_x^2}. \tag{5.16}$$

Noting that $\mathrm{Re}(z_x) < 0$, it can be shown that $|s_x| \le 1$. In a similar manner, for the symbol s_y of the scheme (5.12), $|s_y| \le 1$. It follows that the LOD-ETF scheme described by (5.7) and (5.12) for the convection-diffusion equation (5.1) is unconditionally stable.

6. NUMERICAL EXPERIMENTS

We next consider four test problems to assess the computational performance of the obtained LOD-ETF schemes for the diffusion equation with Dirichlet and Neumann boundary conditions, for a nonlinear reaction-diffusion equation and for the convection-diffusion equation. In each case, we compare its performance with the LOD Crank-Nicolson (LOD-CN) scheme. For all the computations reported in the following, we choose $N = 19$.

PROBLEM 1. We consider the diffusion equation

$$\frac{\partial}{\partial t} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad 0 < x, y < 1, \quad t > 0, \tag{6.1a}$$

with the initial condition

$$u(x, y, 0) = \sin(\pi x)\sin(2\pi y), \tag{6.1b}$$

with the Dirichlet boundary conditions

$$\begin{aligned} u(0, y, t) &= 0, \qquad u(1, y, t) = 0, \\ u(x, 0, t) &= 0, \qquad u(x, 1, t) = 0. \end{aligned} \tag{6.1c}$$

The exact solution is given by

$$u(x, y, t) = e^{-5\pi^2 t}\sin(\pi x)\sin(2\pi y). \tag{6.1d}$$

We solved problem (6.1) with $\Delta t = 0.1$; the maximum absolute errors in the computed solution by the LOD-CN scheme and by the LOD-ETF scheme for different times are shown in Table 1. From the table, it is clear that the LOD-ETF scheme provides much better accuracy for the computed solution.

Table 1. Maximum absolute errors.

t	LOD-CN	LOD-ETF
0.1	1.2(−1)	2.5(−2)
0.2	1.2(−2)	2.8(−4)
0.3	1.3(−3)	6.5(−6)
0.4	1.5(−4)	1.1(−7)

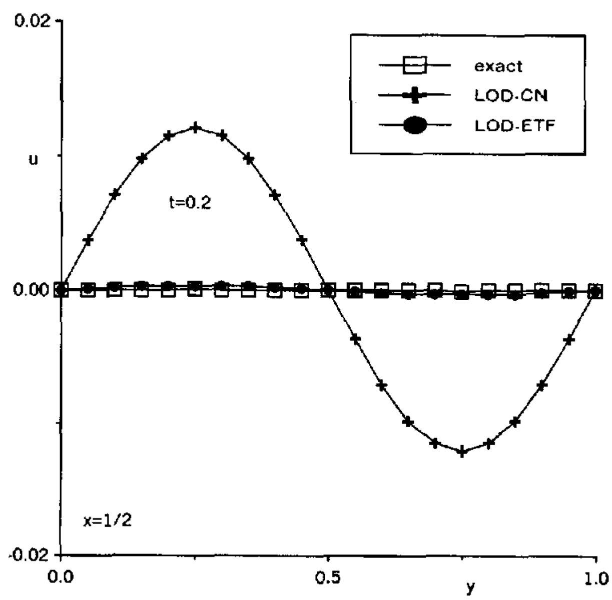

Figure 1. Problem 1.

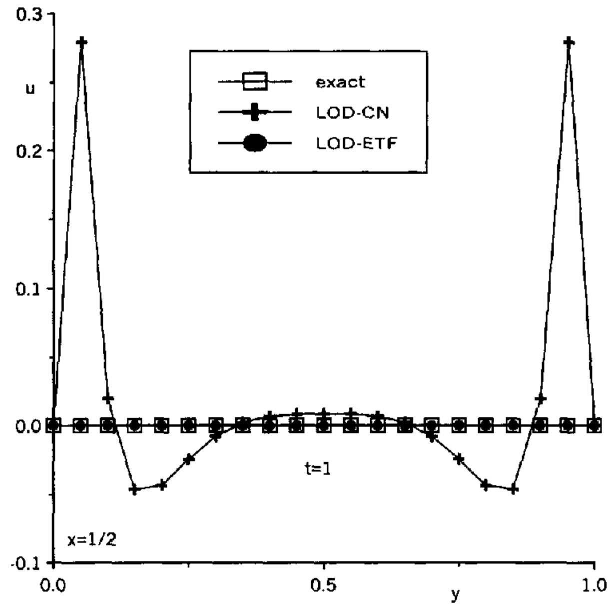

Figure 2. Problem 2.

Table 2. Maximum absolute errors.

t	LOD-CN	LOD-ETF
0.5	4.2(−1)	9.1(−3)
1.0	2.8(−1)	6.6(−5)
1.5	2.0(−1)	4.7(−7)
2.0	1.5(−1)	3.4(−9)

Table 3. Maximum absolute errors.

t	LOD-CN	LOD-ETF
8	3.1(−2)	1.2(−2)
12	3.0(−2)	6.7(−4)
16	2.5(−2)	9.2(−5)
20	2.1(−2)	1.2(−5)

In Figure 1, we show these approximations for the section $x = 1/2$ for time $t = 0.2$; while the LOD-CN approximations oscillate widely, the LOD-ETF approximations give a much better idea of the true solution at this time.

PROBLEM 2. We consider the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad 0 < x, y < 1, \quad t > 0, \tag{6.2a}$$

with the initial condition

$$u(x, y, 0) = 1, \tag{6.2b}$$

with mixed boundary conditions

$$\begin{aligned} \frac{\partial}{\partial x} u(0, y, t) &= 0, \qquad & \frac{\partial}{\partial x} u(1, y, t) &= 0, \\ u(x, 0, t) &= 0, \qquad & u(x, 1, t) &= 0. \end{aligned} \tag{6.2c}$$

The exact solution is given (see [8]) by

$$u(x, y, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-(2n-1)^2 \pi^2 \nu t} \sin((2n-1)\pi y). \tag{6.2d}$$

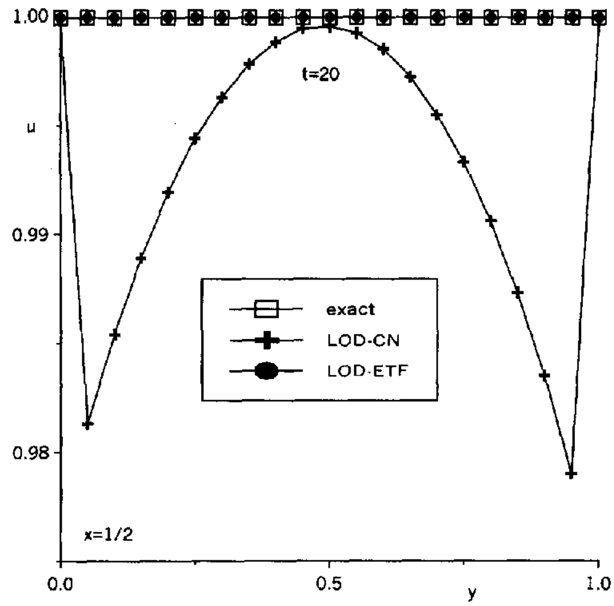

Figure 3. Problem 3.

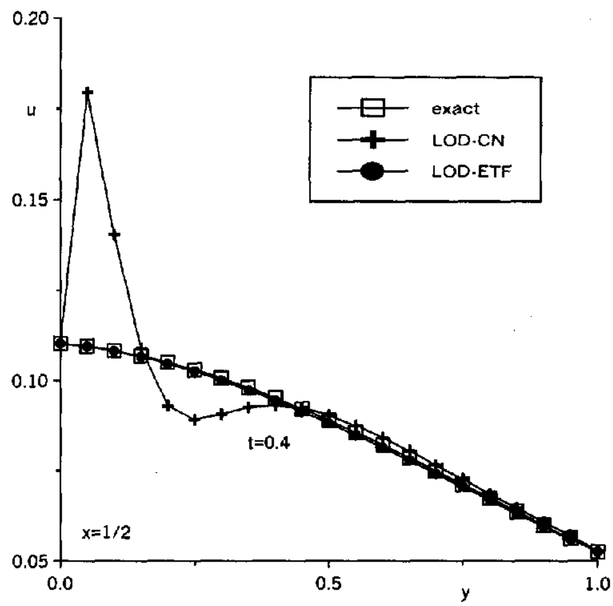

Figure 4. Problem 4.

We solved problem (6.2) with $\Delta t = 0.1$; the maximum absolute errors in the approximations provided by the LOD-CN scheme and the LOD-ETF scheme are shown in Table 2; it is clear that the LOD-ETF scheme provides much superior approximations for the true solution.

In Figure 2, we show these approximations for the section $x = 1/2$ for time $t = 1.0$; while the LOD-CN approximations have wild oscillations, LOD-ETF approximations give almost an exact solution at this time.

PROBLEM 3. We consider the nonlinear reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + u^2(1-u), \qquad 0 < x, y < 1, \quad t > 0, \tag{6.3a}$$

with the initial condition and the boundary conditions taken to be consistent with the exact solution (see [9])

$$u(x,y,t) = \frac{1}{1+e^{p(x+y-pt)}}, \qquad p = \frac{1}{\sqrt{2}}. \tag{6.3b}$$

We computed the solution of problem (6.3) with $\Delta t = 4$; the maximum absolute errors in the approximations provided by the LOD-CN scheme and the LOD-ETF scheme are shown in Table 3. Note that with increasing time, there is a marked improvement in the accuracy of the approximations provided by the LOD-ETF scheme.

In Figure 3, we show these approximations for the section $x = 1/2$ for time $t = 20$. The LOD-CN approximations suffer wild oscillations whereas the LOD-ETF approximations are right on the spot.

PROBLEM 4. We consider the convection-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad 0 < x, y < 1, \quad t > 0, \tag{6.4a}$$

with the initial condition and the boundary conditions taken to be consistent with the exact solution (see [9])

$$u(x,y,t) = \frac{1}{\sqrt{s}} \exp\left(-50(x+y-t)^2/s\right), \qquad s = 1 + 200t. \tag{6.4b}$$

We computed the solution of problem (6.4) with $\Delta t = 0.2$; the maximum absolute errors in the approximations provided by the LOD-CN scheme and the LOD-ETF scheme are shown in Table 4. Right from the beginning, the LOD-ETF scheme provides much better accuracy which improves with increasing time.

Table 4. Maximum absolute errors.

t	LOD-CN	LOD-ETF
0.2	2.4(−1)	7.5(−3)
0.4	1.1(−1)	9.1(−4)
0.6	6.8(−2)	1.8(−4)
0.8	3.7(−2)	6.2(−5)

In Figure 4 we show these approximations for the section $x = 1/2$ for time $t = 0.4$. The LOD-CN approximations have large oscillations in the first-half of the y-interval, while the LOD-ETF gives a consistently satisfactory approximation for the true solution.

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